Algebra 1 Prelims: August 2020

1) Assume G acts on a finite set S in such a way so there is ***only one orbit***. Let $s \in S$, and let $G_s = \{g \in G \mid gs = s\}$ be the isotropy subgroup of s *(sometimes called the stabilizer of s)*.

a) Give a formula for $|S|$, the cardinality of S, and prove that your answer is correct.

b) If $s' = gs$ for some $g \in G$, state the relation between G_s and $G_{s'}$ and briefly justify your answer.

c) Suppose $\{g \in G \mid gx = x\ \forall x \in S\} = \{e\}$. Show if N is any normal subgroup of G that is contained in G_s for one particular s, then $N = \{e\}$.

2,3) Answer two of the following three problems (***A, B, C***). Note that each of these problems has multiple parts *(if you attempt all three, I will count your best 2)*.

Ai) If N is a normal subgroup of G containing no elements of the commutator subgroup G^c except the identity, show $N \subseteq Z(G)$ where $Z(G)$ denotes the center of G.

ii) Suppose G is a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G must be abelian and justify all of your steps.

Bi) Show that if G is any group of order p^2q for distinct primes p, q then at least one of its Sylow subgroups must be normal.

ii) Let G be a finite group, H a normal subgroup, and P a Sylow p-subgroup of G for some prime p. Prove that $P \cap H$ is a Sylow p-subgroup of H.

Would this still be true if H were not normal? Prove if true, and if false, explain where your proof would break down and also give a counterexample or explain how you know that a counterexample must exist

C) Let $\mathcal{M}$ be a $\mathbb{Z}$-module *(i.e. an abelian group under addition)* and let $\mathcal{S} \subseteq \mathcal{M}$ be a proper subset.

i) For each of the following two statements a and b, tell whether a implies b and also whether b implies a. In each case, if the implication is true, prove it, and if the implication is false, give a counterexample.

a) $\mathcal{S}$ spans $\mathcal{M}$ as a $\mathbb{Z}$-module and no proper subset of $\mathcal{S}$ has this property.

b) $\mathcal{S}$ is linearly independent over $\mathbb{Z}$ and no set properly containing $\mathcal{S}$ has this property.

ii) Let H, K, and J be *(possibly infinite)* abelian groups, and suppose that $J \times H \approx K \times H$. Does this imply that $J \approx K$? Prove it or give a counterexample.

4) Recall that an element x of a ring A is called nilpotent if $x^n = 0$ for some $n \in \mathbb{Z}^+$.

a) Suppose A is a commutative ring with $1 \neq 0$ satisfying the property that the localization A_m has no nonzero nilpotent elements for any maximal ideal m. Prove that A has no nonzero nilpotent elements.

b) Would the analogous result still be true if the phrase "no nonzero nilpotent elements" was replaced by the phrase "no zero-divisors", *(where by definition a zero divisor can not equal 0)*? If your answer is yes, give a complete proof, and if your answer is no, tell where your proof in part a breaks down, and also give a counterexample.

5) Assume all rings in this problem are commutative. Use the maximality property of Noetherian rings to prove each of the following.

a) In a Noetherian ring, any ideal contains a product of prime ideals.

b) In a Noetherian ring, any ideal can be written as the intersection of a finite number of irreducible ideals *(where an ideal in a commutative ring is said to be irreducible if it can* **not** *be written as the intersection of two properly bigger ideals)*.