ALGEBRA I PRELIM, DECEMBER 2013
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(ANALYSIS AND TOPOLOGY PRELIMS FOLLOW)
1a) If \( f : G \rightarrow G' \) is a group homomorphism and \( H \) is a subgroup of \( G \), is it always the case that \( f^{-1}(f(H)) = H \) (where \( f^{-1}(f(H)) \) denotes the inverse image of \( f(H) \) in \( G \) or equivalently the set of all elements in \( G \) that map into \( f(H) \))? Prove it or give a counterexample.

b) If the answer to part a is no, then give a complete description of the subgroup of \( G \) that equals \( f^{-1}(f(H)) \) and apply this description to your counterexample. If the answer to part a is yes, then you can skip this part of the question.

2) Let \( G^c \) denote the commutator subgroup of a group \( G \).

a) Let \( N \) be any normal subgroup of \( G \). Prove that \( G/N \) is abelian if and only if \( G^c \subseteq N \).

b) Show if \( H \) is any subgroup of \( G \) such that \( G^c \subseteq H \), then \( H \) is a normal subgroup of \( G \).

c) Recall that a group \( G \) is called solvable if there exists a normal tower of subgroups starting with \( G \) and ending with \{ \( e \) \} for which all the factor groups are abelian. Now define the following subgroups of \( G \), \( G^{(0)} = G \), \( G^{(1)} = G^c = \text{commutator subgroup of } G \), \( G^{(2)} = (G^{(1)})^c = \text{commutator subgroup of } G^{(1)} \), \( G^{(3)} = (G^{(2)})^c \), etc. Show that \( G \) is solvable if and only if \( G^{(s)} = \{ e \} \) for some positive integer \( s \).

d) Suppose that \( K \triangleleft H \triangleleft G \) and \( K \) is a Sylow \( p \)-subgroup of \( H \) for some prime \( p \), prove that \( K \triangleleft G \).

b) Suppose \( G \) is a finite group, \( P \) is a Sylow \( p \)-subgroup, and \( H \) a (not necessarily Sylow) \( p \)-subgroup of \( G \). If \( H \subseteq N_G(P) \), prove that \( H \subseteq P \).

4) Let \( A \) be an integral domain and \( M \) an \( A \)-module. An element \( x \in M \) is called a torsion element of \( M \) if the annihilator of \( x \), \( \text{Ann}(x) \), does not equal \{ \( 0 \) \}.

a) Show that the subset \( T(M) \) of torsion elements of \( M \) forms a submodule of \( M \). Would this still be the case, if \( A \) were not an integral domain? Prove it or give a counterexample.

b) If \( f : M \rightarrow N \) is a homomorphism of \( A \)-modules, show that \( f(T(M)) \subseteq T(N) \).

c) If \( 0 \rightarrow M' \xrightarrow{f'} M \xrightarrow{g} M'' \) is an exact sequence of \( A \)-modules, show that \( 0 \rightarrow T(M') \xrightarrow{f'} T(M) \xrightarrow{g'} T(M'') \) is also exact where \( f' \) and \( g' \) are the restrictions of \( f \) and \( g \) to \( T(M') \) and \( T(M) \), respectively.

d) Would the result in (c) still be true if we added a 0 to the right-hand side of the above two sequences? Prove it or give a counterexample.

5) For each of the following, give an example if possible, and if not possible, briefly explain why not. All rings in this problem are assumed to be commutative and all of your examples should involve only commutative rings. For purposes of the prelims, you must get at least 5 of these correct in order to have the problem count as being correct.

(a) A commutative ring \( A \) and a multiplicatively closed subset \( S \) which does not contain \( 0 \), such that \( S^{-1}(A) \) is not a local ring.

(b) A commutative ring \( A \) and a multiplicatively closed subset \( S \) which does not contain \( 0 \), such that all prime ideals of \( A \) intersect \( S \).

(c) A subring \( A \) of a Noetherian ring \( B \) that is not a Noetherian ring.

(d) A ring \( A \) and a submodule of a finitely generated \( A \)-module that is not finitely generated.

(e) A non-Noetherian ring \( A \) that is a Noetherian \( \mathbb{Z} \)-module.

(f) A Noetherian ring \( A \) that is a non Noetherian \( \mathbb{Z} \)-module.
Problem #1: Let \( f : [0, 1] \to \mathbb{R} \).

i) Define what it means for \( f \) to be Lipschitz. Also define what it means for \( f \) to be absolutely continuous.

ii) Prove that if \( f \) is Lipschitz, then it is absolutely continuous.

iii) Prove that \( f(x) = \sqrt{x} \) is absolutely continuous, but not Lipschitz on \([0, 1]\).

Problem #2: Define what it means for a function \( f : [a, b] \to \mathbb{R} \) to be of bounded variation. Prove that a function of bounded variation is a difference of two monotone functions.

Problem #3: Recall that \( E \subset \mathbb{R}^d \) is Lebesgue measurable if for every \( \epsilon > 0 \) there exists an open set \( O \) containing \( E \) such that \( m^*(O\setminus E) \leq \epsilon \), where \( m^* \) is the outer measure. Prove that \( E \subset \mathbb{R}^d \) is Lebesgue measurable if and only if for every \( A \subset \mathbb{R}^d \),

\[
m^*(A) = m^*(A \cap E) + m^*(A \setminus E).
\]

Problem #4: Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function on \([a, b]\) and twice continuously differentiable on \((a, b)\).

i) Suppose that \( f''(x) > 0 \) on \((a, b)\). Prove that

\[
f((1-t)a + tb) \leq (1-t)f(a) + tf(b).
\]

ii) Use i) to show that if \( x, y \geq 0 \) and \( \frac{1}{p} + \frac{1}{q} = 1, p > 1 \), then

\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q}.
\]

iii) Use ii) to show that if \( f, g \) are absolutely integrable on \( \mathbb{R} \), then

\[
\int f(x)g(x)dx \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \cdot \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}.
\]
Problem #5: Let \( f : \mathbb{R}^d \to \mathbb{C} \) be an absolutely integrable function. Prove that
\[
m \left( \left\{ x \in \mathbb{R}^d : \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy > \lambda \right\} \right) \leq \frac{C}{\lambda} \| f \|_{L^1(\mathbb{R}^d)},
\]
where \( m \) is the Lebesgue measure on \( \mathbb{R}^d \) and the supremum is taken over all the dyadic cubes \( Q \). Recall that a dyadic cube is a cube of the form
\[
\left[ \frac{i_1}{2^n}, \frac{i_1 + 1}{2^n} \right] \times \ldots \times \left[ \frac{i_d}{2^n}, \frac{i_d + 1}{2^n} \right]
\]
for some integers \( n, i_1, \ldots, i_d \).

Hint: Follow the proof of the Hardy-Littlewood maximal inequality. You may find special properties of dyadic cubes helpful in establishing an appropriate covering lemma.
I. Let $A$ be a set. Let $\{X_\alpha\}_{\alpha \in J}$ be a indexed family of spaces, and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f : A \rightarrow X_\alpha$.

(a) Show that in a unique smallest (coarsest) topology $\mathcal{T}$ on $A$ such that each $f_\alpha$ is continuous.

(b) Let $S_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\}$ and $S = \cup_\beta S_\beta$. Show that $S$ is a subbasis for the topology $\mathcal{T}$ on $A$.

(c) Show that a map $g : Y \rightarrow A$ is continuous relative to $\mathcal{T}$ if and only if each composition $f_\alpha \circ g$ is continuous.

(d) Let $f : A \rightarrow \prod_{\alpha \in J} X_\alpha$ be defined by

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

and $Z$ be the subspace $f(A)$ of the product space $\prod_\alpha X_\alpha$. Show that the image under $f$ of each element of $\mathcal{T}$ is an open subset of $Z$.

II. Let $X$ be a connected topological space. Let $x \in X$ be such that $X - \{x\}$ is disconnected with separation $X = A \cup B$. Prove $A \cup \{x\}$ is connected.

III. A topological space $X$ is countably compact if every countable open covering of $X$ contains a finite subcollection that covers $X$. Assume $X$ is a Hausdorff space. Prove that the countably compact condition is equivalent to the (Bolzano-Weierstrass) condition that every infinite set in $X$ has a limit point.

IV. Recall that a Baire space is characterized by the condition that any countable union of closed sets, each with empty interior, has empty interior. Prove that an open subspace of a Baire space is a Baire space.

V. Let $\{A_\alpha\}$ be a locally finite collection of subsets of a topological space $X$. This condition means that each point of $X$ lies in an open set which has a non-empty intersection with at most finitely many of the $A_\alpha$. Prove

$$\bigcup_\alpha A_\alpha = \bigcup_\alpha \overline{A_\alpha}.$$