Agenda: Complex Processes in Nature and Laboratory

Systems and dynamics, qualifiers
    Examples (climate, planetary motion),

Order and Chaos, determinism and stochastic unpredictability
    1D dynamics: phase space curves/orbits

Non-linear dynamics in nature and their modeling
    mathematical model (climate, logistic map)
    Stability criteria, stationary states

Self replicating structures out of simplicity
    Cellular automata and fractal structures,
    Self-organization in coupled chemical reactions

Thermodynamic states and their transformations
    Collective and chaotic multi-dimensional systems
    Energy types equilibration,
    flow of heat and radiation

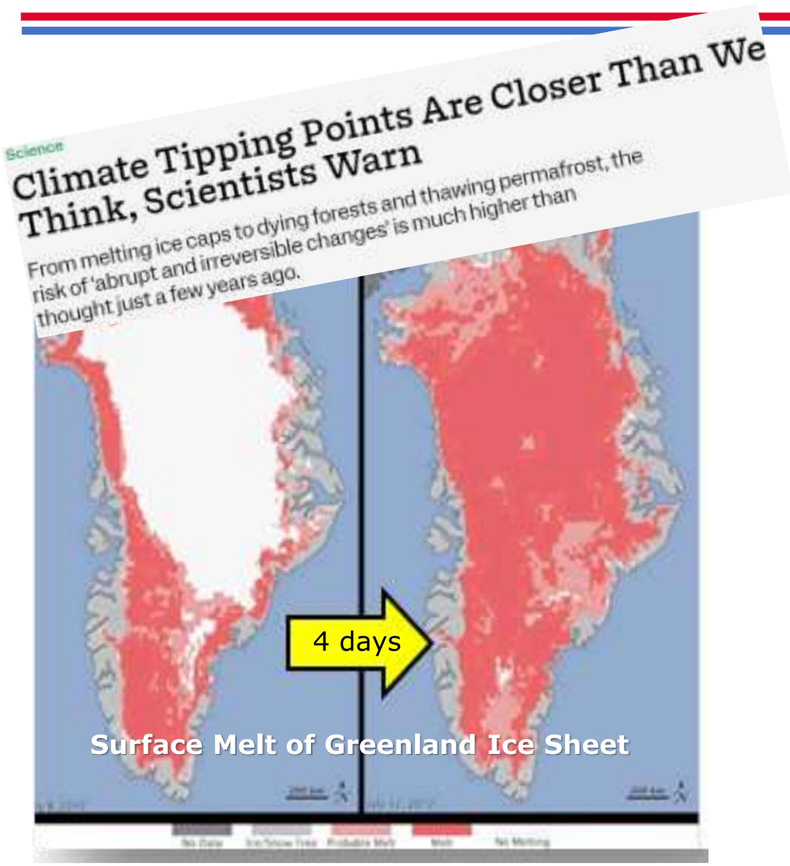
Reading Assignments
Weeks 1&2

LN II: Complex processes

Kondepudi Ch.19
Additional Material
J.L. Schiff:
Cellular Automata,
Ch.1, Ch. 3.1-3.6

McQuarrie & Simon
Math Chapters
MC B, C, D,

Tipping Points in Earth Climate ?



Non-linear and coupled effects in Earth current climate evolution → global warming, melting of sea ice , ice cap, desertification, ocean acidification, sea level rise,……

<u>Historic climate facts:</u>

Earth climate has alternated between Ice ages (little and major) and greenhouse periods. Transition speed?

Do we have time to adapt or change pace?

Mind the fate of planet Venus (NYT 012921)

Earth albedo or surface reflectivity ε = important in maintaining radiation balance

Glaciation: increasing ice cover $\Delta\varepsilon > 0 \rightarrow$ *surface temperature change* $\Delta T < 0$

Warming: decreasing ice cover $\Delta\varepsilon < 0 \rightarrow$ *surface temperature change* $\Delta T > 0$

Albedo is non-monotonic function of important driving parameters, has extrema!

Earth Albedo Model

Albedo is non-monotonic function of important driving parameters.

Combine ε parameter dependence to model *non-linear* dependence on history:

$$\varepsilon\left(t+\Delta t\right)=\alpha\cdot\varepsilon\left(t\right)-\beta\cdot\varepsilon^{2}\left(t\right)+......;\quad parameters\ \alpha,\beta=f(CO_{2},...)?$$

$$Since\ \varepsilon\left(t\right)\ is\ non-monotonic\ and\ must\ have\ an\ extremum$$

$$\rightarrow sign\left(\alpha\right)=sign\left(\beta\right),\ choose\ \alpha,\beta>0$$

$$Adopt\ discrete\ time\ steps\ t_{n}\ \ (days,\ months,\ years,....,centuries)\rightarrow$$

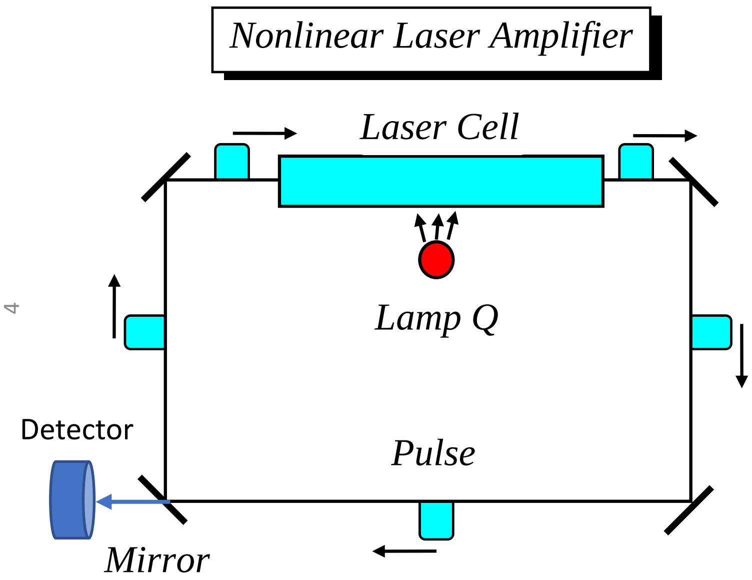
$$\varepsilon_{n+1}=\varepsilon_{n}\left(t+n\cdot\Delta t\right)\approx\alpha\cdot\varepsilon_{n}-\beta\cdot\varepsilon_{n}^{2}\qquad "Iteration"$$

$$Variable\ transformation\rightarrow$$

$$Profile\ function\ \boxed{f(\varepsilon)=\mu\cdot\varepsilon\cdot\left(1-\varepsilon\right)}\quad "Logistic\ Map"$$

$$\varepsilon_{n+1}=f(\varepsilon_{n})=f(f(\varepsilon_{n-1}))=f(f(f(\varepsilon_{n-2})))=f^{3}(\varepsilon_{n-2})\quad Iterative\ Logistic\ Map$$

Laboratory Experiments On Complex (Chaotic) Dynamics



To investigate expected behavior of physical system → study mathematical properties of profile function and associated maps.

→ Test with laboratory experiments.

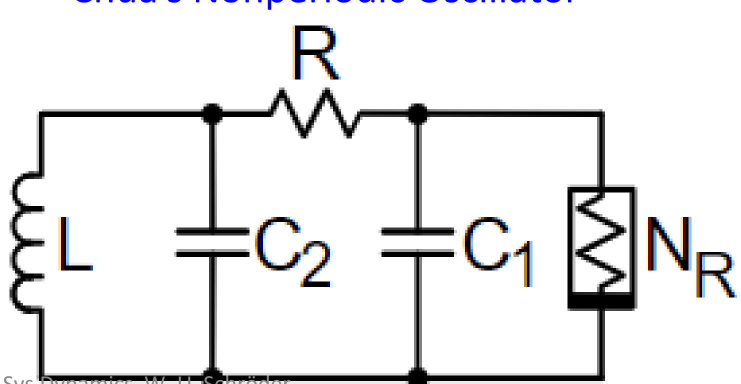
Initial maximum laser cavity intensity $I = 1$
Once around the track → $I_0 < 1$ → cavity
Stimulated emission ∝ **product of**
trigger intensity **x** available inversion

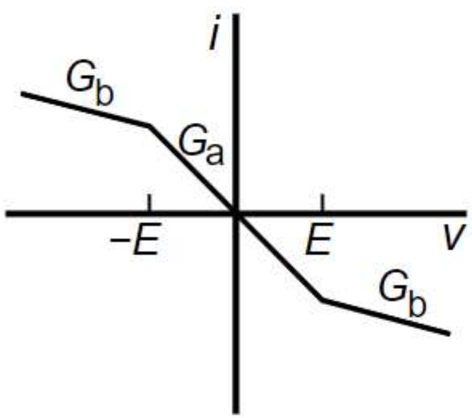
$$I_1 = \mu \cdot I_0 \cdot (1 - I_0) \quad etc\ n > 0$$

Logistic Map

n = number of circuits completed

Chua's Nonperiodic Oscillator

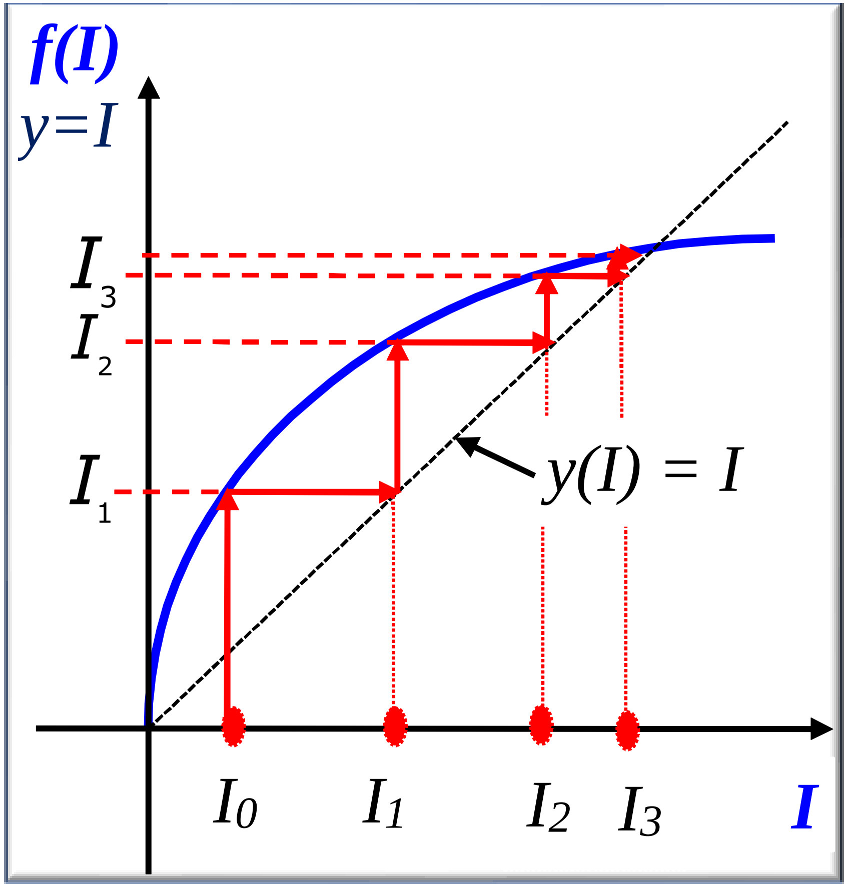




Chua Diode N_R : nonlinear negative resistance = amplifier with positive feedback.

Graphing An Iteration ("Cobweb Plot")

Sequence $I, f(I), f^2(I), \ldots, f^n(I) \ldots$
Plotted in 2D : $f(I_n)$ vs. I_n

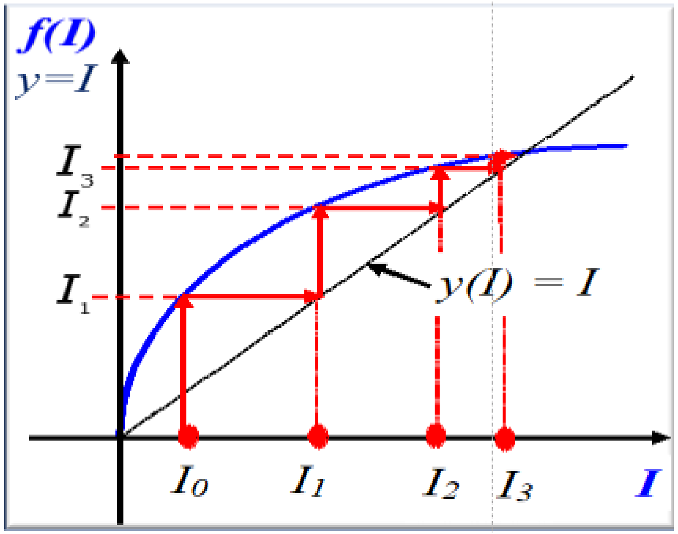


1. Draw horizontal (***I***) and vertical (***f***) axes of a 2D Cartesian coordinate system, with equal divisions.
2. Plot the map profile function *f(I) vs.* ***I .***
3. Plot the diagonal line ***y(I)*** = ***I.***

4. Start drawing the trajectory I_n, ($n = 0, 1\ldots\ldots$) by marking the initial point $I_{n=0}$ on the horizontal axis.
5. Draw a vertical arrow, from point I_n, to its functional value $I_{n+1} = f(I_n)$ on the profile curve.
6. Draw a horizontal arrow from point $f(I_n)$ to the point $f(I_n) = I_n$ on the $y = I$ line. This identifies the abscissa coordinate I_n for the next iteration.

7. Go to *5*) and repeat *5*) and *6*) until done.

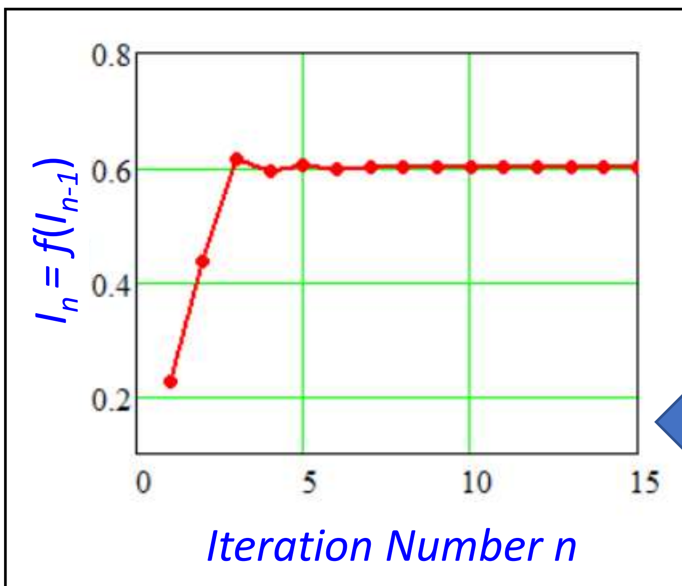
Graphing An Iteration II



Sequence I, $f(I)$, $f^2(I)$,...,$f^n(I)$...
Plotted in 2D

$f(I_n)$ vs. I_n

Different In : Laser intensity flickers

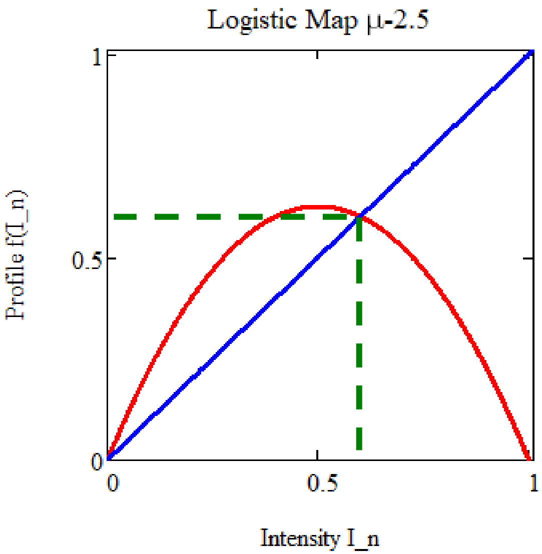


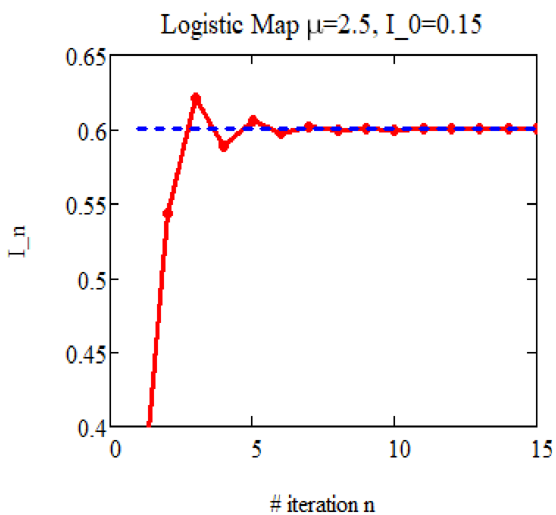
Sequence I, $f(I)$, $f^2(I)$,...,$f^n(I)$...
Plotted in 1D *vs.* I

Intensity I_n *vs. Iteration number n*

Intensity increases at first, then oscillates slightly. Finally, gets to steady-state operation after a few initial circuits (periods).

Logistic Map Features





Features of an iteration on a map depend on the profile function *f*, specifically on the amplification factor μ and the initial conditions,
InCon for 1D: just the starting point I_0.

$$Periodic\ point\ I_{pm},\ period\ m: f^{m}(I_{pm}) = I_{pm}$$

$$Fixpoints\ I_f : f(I_f) = I_f \qquad Trivial\ I_f = 0$$

$$Non-trivial\ FP\ exist\ if\ f(I)\ and\ y(I) = I\ intersect$$

$$\boxed{Condition\ \ I_f = \frac{\mu - 1}{\mu} \geq 0}$$

$$Trajectory\ ensembles\ with\ I_0 \approx I_f$$

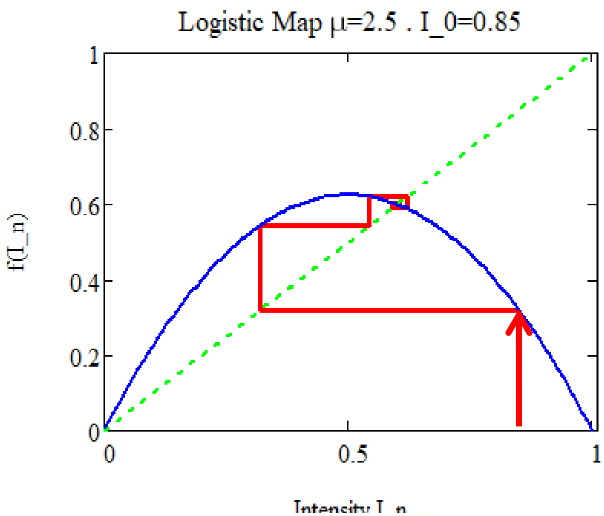
$$fixpoints\ "attract"\ or\ "repel"\ (scatter)$$

$$\left|\left(\frac{df}{dI}\right)_{I_f}\right| < 1 \quad (I_f = Attractor) \qquad \left|\left(\frac{df}{dI}\right)_{I_f}\right| > 1 \quad (I_f = Repellor)$$

$$\left|\left(\frac{df}{dI}\right)_{I_f}\right| = 0 \quad (I_f = ???)$$

Chaotic behavior if sensitivity to initial condition.

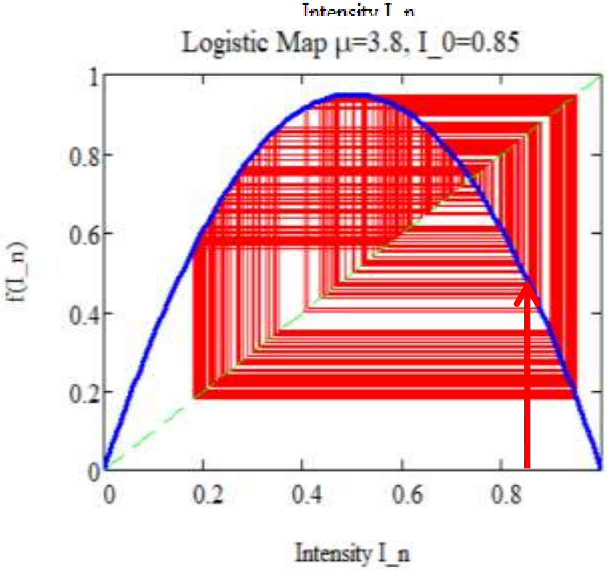
Order and Chaos Parameter Dependence

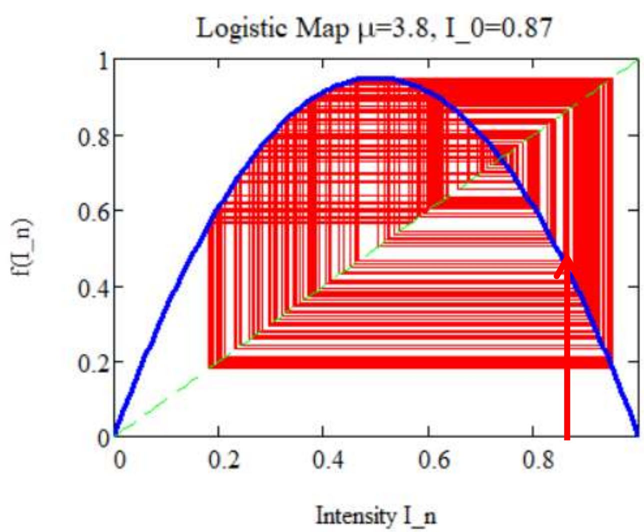


μ = *2.5:* Fixpoint = attractor. All trajectories end up in this point: Laser operation stable after startup.

μ = *3.8* Fixpoint = strange attractor.
Trajectories spiral initially around fixpoint: intensity blinks slightly. After a few cycles, oscillations between 3 and 4 different brightness levels, highly unstable, essentially right after start.

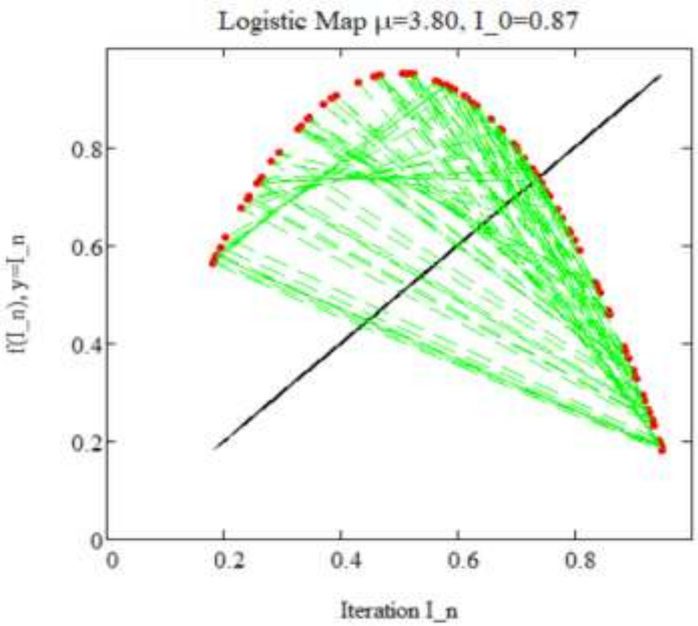
Sensitivity to initial conditions → chaotic operation





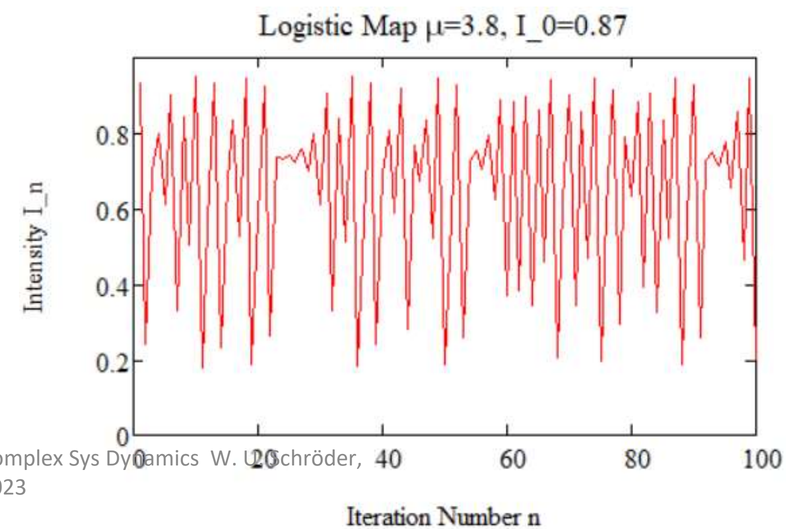
Slightly different I_0 lead to very different time behavior.
N=500 iterations

Chaotic Map Trajectories



Same example as above, plot showing only the iterative intensities I_n on the curve representing the map profile function $f(I)$.

A large part of the brightness spectrum is covered by the trajectory already after 500 iteration. No apparent intensity pattern.
Intensity flashes between bright and dim.



Same example as above, plot shows iterative intensities I_n vs n. Some, but not exact similarities, intermittency domains, strongly dependent on initial condition I_0.

Sensitivity to Initial Conditions

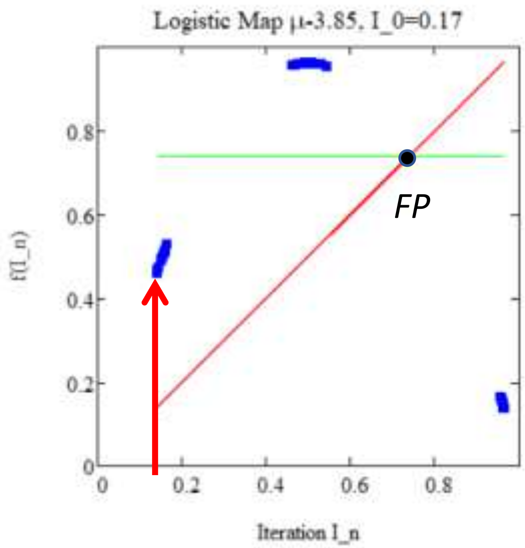


Illustration of sensitivity to initial conditions for
μ = 3.85, fixpoint at *I = 0.74, strange attractor*
IC: I_0 = 0.17, *N* = 100 iterations
Blinking alternatively with 3 different intensities

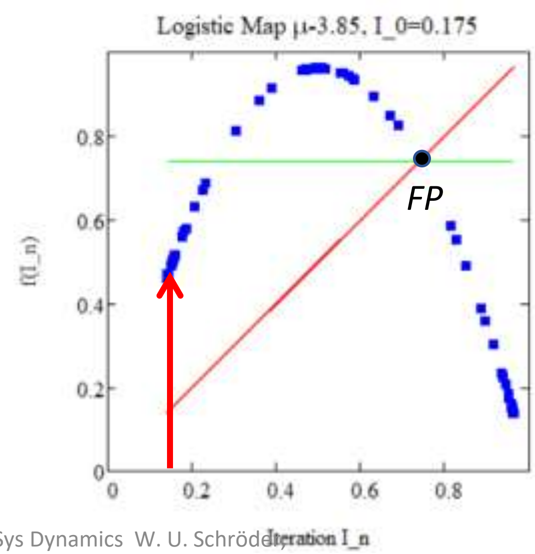
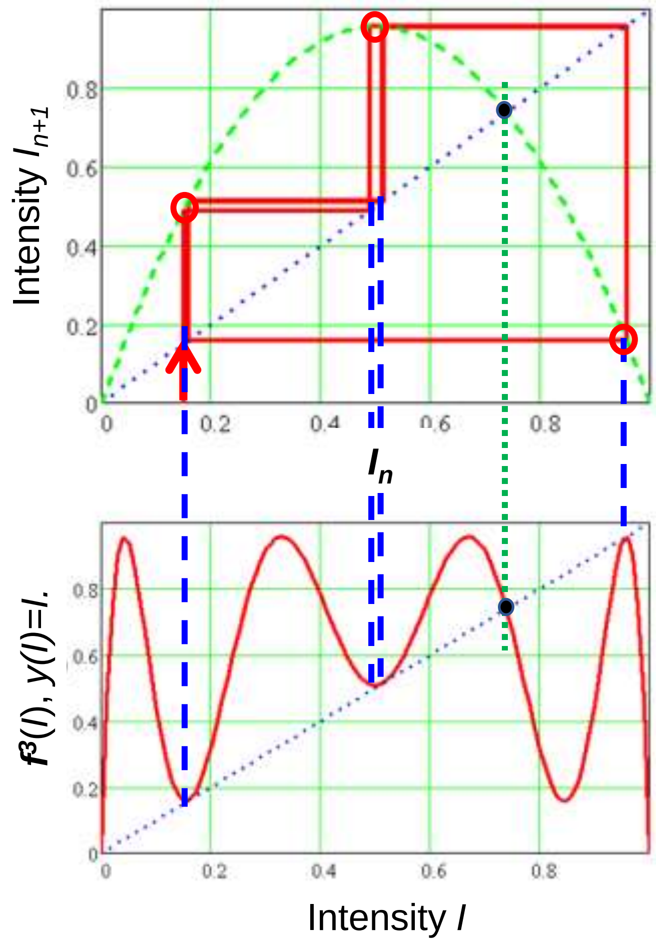


Illustration of sensitivity to initial conditions for
μ = 3.85, fixpoint at *I = 0.74, strange attractor*
IC: I_0 = 0.175, *N* = 100 iterations
Blinking alternatively with a continuum of intensities filling most of the accessible intensity range

Periodic Flashes

Metastable/intermittent processes, strange but predictable trajectories: search for "periodic points." Points of period n = stable (attractor) fixpoints of $\boldsymbol{f}^{\mathbf{n}}$(x).

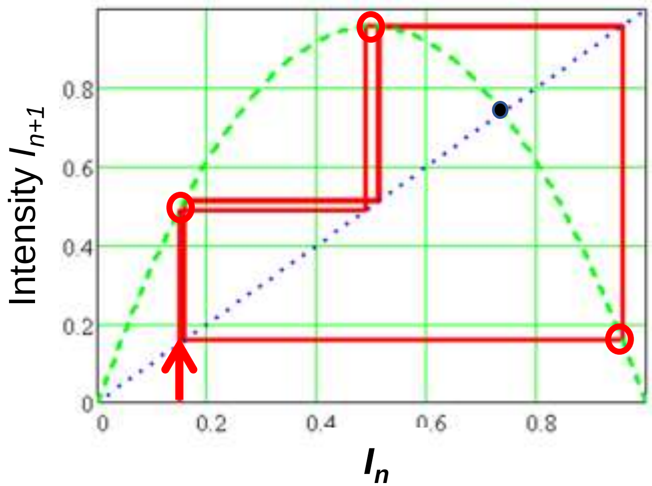


Fixpoint at I_f = 0.653 (black dot)
= "strange" attractor:
Trajectory cycles around $\boldsymbol{I}_f$ in *3* periods.

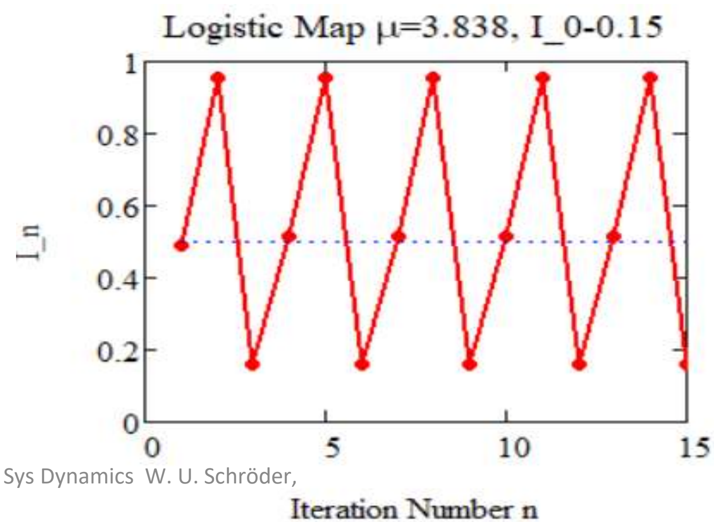
Finding members of strange cycle: look for tangential touching of curve
$\boldsymbol{f}^3(\mu, \boldsymbol{I})$ at $y(I)=I$.

Periodic Flashes

Metastable/intermittent processes, strange but predictable trajectories: search for "periodic points." Points of period n = stable (attractor) fixpoints of $\boldsymbol{f^n}(x)$.



Fixpoint at I_f = 0.653 (black dot) = "strange" attractor:
Trajectory cycles around $\boldsymbol{I}_f$ in *3* periods.



Pattern $\boldsymbol{f}(\mu, \boldsymbol{I})$ exhibiting periodic triplet blinking patterns : medium, high, low intensity.
Deterministic

Linear and Non-Linear Dynamical Regimes

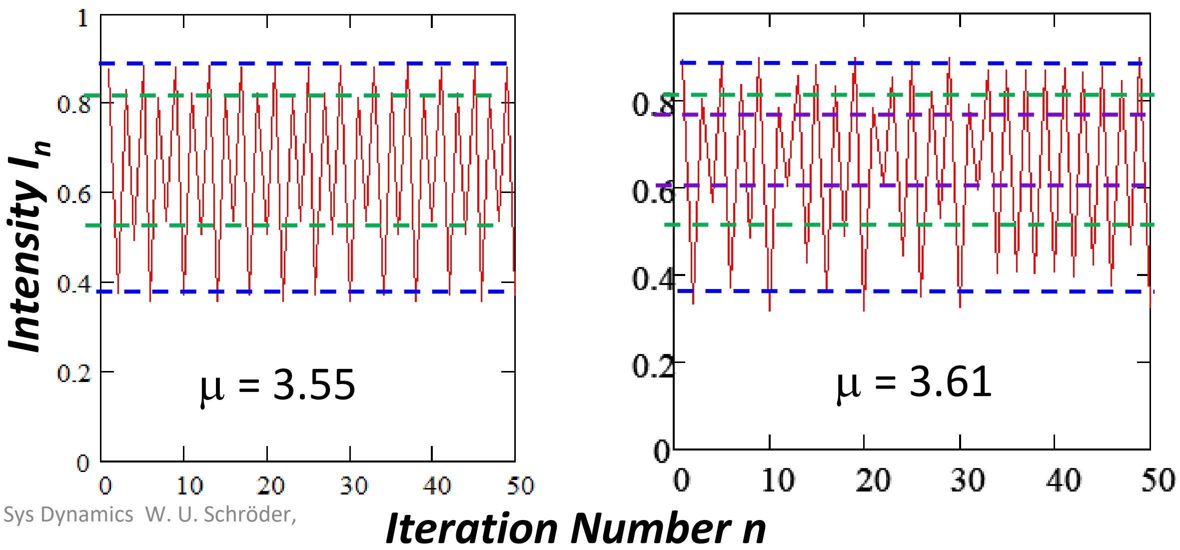
$0.0 \le \mu \le 1.0$: *No non – trivial fixpoints* $\rightarrow I_n \xrightarrow[n\to\infty]{} 0$

$1.0 < \mu \le 3.0$: *1 non – trivial attractor fixpoint, "deterministic chaos"*
Trajectory deterministic for precise initial condition

$3.0 < \mu \le 3.6$: *1 non – trivial repellor fixpoint, "deterministic chaos"*
bi – stable flickering with alternating intensities,
several n – frequency doublings (bifurcations)

$3.6 < \mu < 3.8$: *1 non – trivial repellor fixpoint, intermittent flicker*

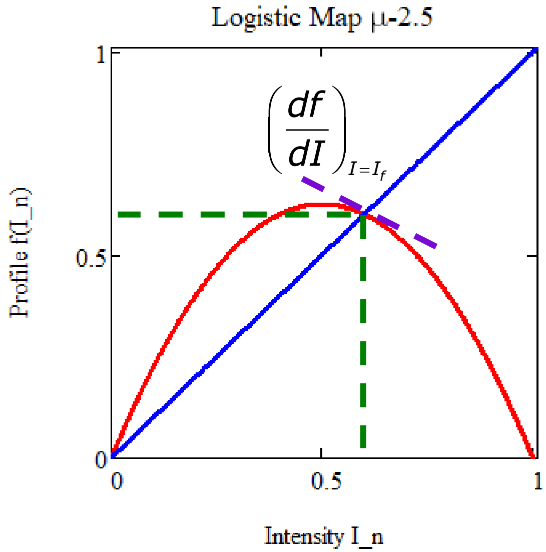
$3.8 \le \mu < 4.0$: *1 non – trivial repellor fixpoint, chaotic dynamics*

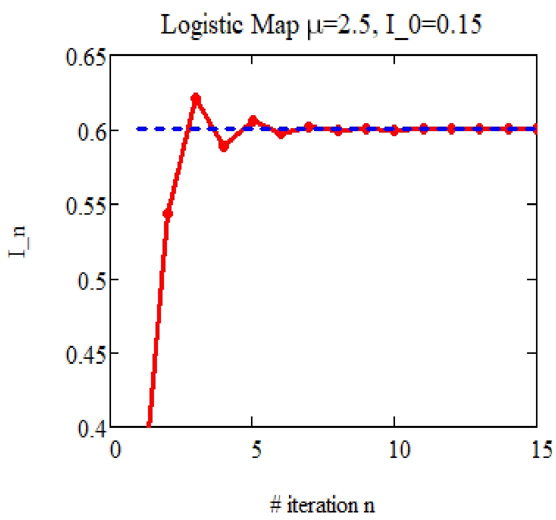


Left: Frequency doubling

Right: Two frequency doublings with intermittency.

Logistic Map Features





Profile function *f*, amplification factor $\boldsymbol{\mu}$

Fixpoints $I_f : f(I_f) = I_f$ *Trivial* $I_f = 0$

Non – trivial FP exists for $\mu > 1$

Trajectory ensembles with $I_0 \approx I_f$

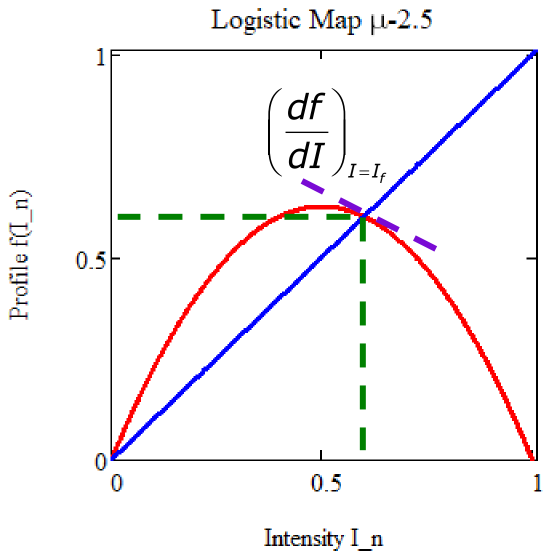
fixpoints "attract" or "repel" (scatter)

$$\left|\left(\frac{df}{dI}\right)_{I_f}\right| < 1 \quad (I_f = \text{Attractor})$$

$$\left|\left(\frac{df}{dI}\right)_{I_f}\right| > 1 \quad (I_f = \text{Repellor, strange attractor})$$

Can you give some plausible geometrical or analytical arguments for this rule?

Logistic Map Features



Profile function *f*, amplification factor $\boldsymbol{\mu}$

$Fixpoints\ I_f : f(I_f) = I_f$ $\quad$ $Trivial\ I_f = 0$

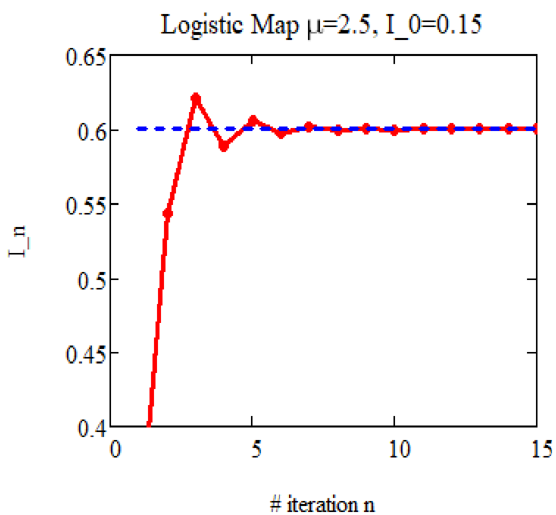
$Non-trivial\ FP\ exists\ for\ \mu > 1$

$Trajectory\ ensembles\ with\ I_0 \approx I_f$

$fixpoints\ "attract"\ or\ "repel"\ (scatter)$

$$\left|\left(\frac{df}{dI}\right)_{I_f}\right| < 1 \quad (I_f = Attractor)$$

$$\left|\left(\frac{df}{dI}\right)_{I_f}\right| > 1 \quad (I_f = Repellor,\ strange\ attractor)$$



Check behavior by varying initial conditions,
Compare trajectories with ($\boldsymbol{I_0 = I_f} \pm \varepsilon$)
→ Different sensitivity to initial condition.
df > d***I*** → *distance between trajectories grows*

Stability of Complex Systems

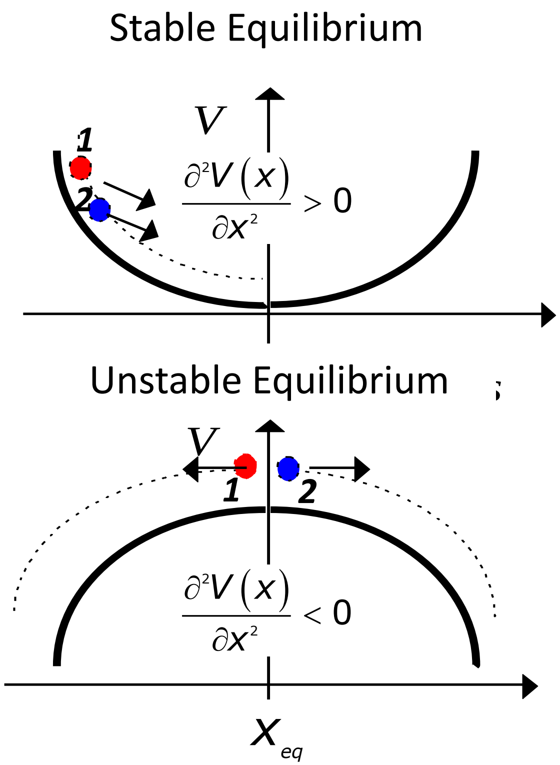


Illustration of potential equilibrium points and trends of neighboring trajectories

What are asymptotic states reached in limit $t, n \rightarrow \infty$?
Can they be reached from any initial conditions?

Specifically: deterministic or chaotic behavior?

→ Need stability criterion,
one-dimensional classical mechanics:

motion driven by a potential $V(x)$

Force equilibrium ←→ $V(x)$=extremum:

$$\left.\frac{\partial V(x)}{\partial x}\right|_{x_{eq}} = 0 \qquad \left.\vec{\nabla} V(\vec{r})\right|_{\vec{r}_{eq}} = \vec{0}$$

Corresponding effects of development of neighboring trajectories:

Converge towards stable equilibrium
Diverge away from unstable equilibrium

Stability of Complex Systems

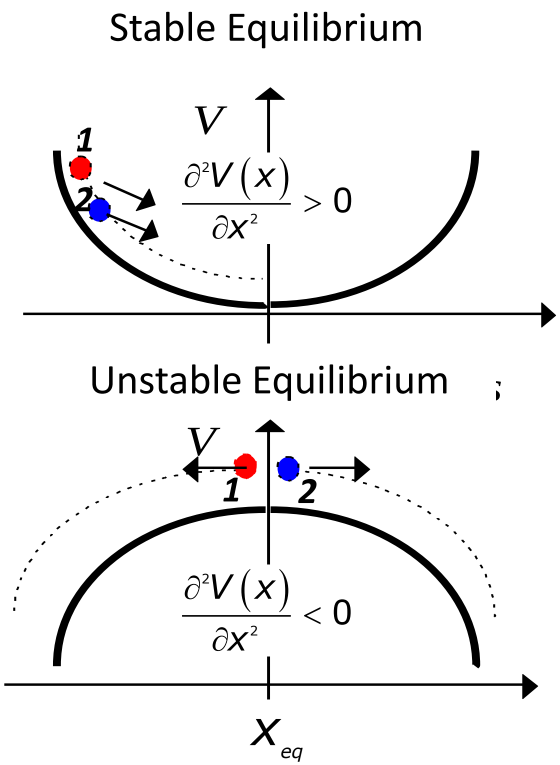


Illustration of potential equilibrium points and trends of neighboring trajectories

Integrate ***1D*** equation of motion *EoM* along ***x*** numerically → *1D* map $\boldsymbol{x_{n+1} = f(x_n)}$

Example: Point particles, mass *m*, force *F*

(Can you write down EoM $\boldsymbol{x}_n = \boldsymbol{x}(t_n)$?)

2 similar initial conditions given ***x*** and ($\boldsymbol{x+\varepsilon}$) small $\varepsilon > 0$.

Step ***n***: trajectories at $f^n(x)$ and $f^n(x+\varepsilon)$

Convergence/divergence ←→ Distance criterion δ

How far apart are initially close trajectories after step ***n?***

$$\delta(\varepsilon, n) := | f^n(x) - f^n(x+\varepsilon) | =: | \varepsilon | \cdot e^{\lambda \cdot n}$$

Legitimate definition of λ, illustrates behavior $n \to \infty$

Lyapunov Stability Criterion

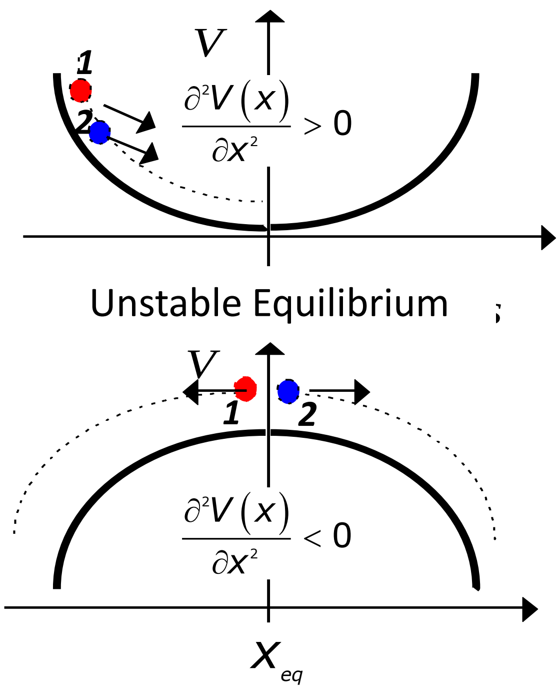


Illustration of potential equilibrium points and trends of neighboring trajectories

Lyapunov exponent:

divergence $\lambda > 0$ Convergence $\lambda < 0$

Large positive exponents indicate extreme sensitivity to initial conditions→ chaotic dynamics

$$\delta(\varepsilon, n) := |f^n(x) - f^n(x+\varepsilon)| =: |\varepsilon| \cdot e^{\lambda \cdot n}$$

$$\rightarrow Ln\left|\left\{\frac{f^n(x) - f^n(x+\varepsilon)}{\varepsilon}\right\}\right| = \lambda \cdot n$$

Infinitesimal ε

$$\boxed{\lambda = \frac{1}{n} Ln\left|\frac{df^n(x)}{dx}\right|}$$

How to calculate derivative of

Implicit function $f^n(x)$?

Lyapunov Exponent

$$\boldsymbol{\lambda} = \frac{1}{n} Ln\left|\frac{df^n(x)}{dx}\right|$$

Implicit function

$$f^n(x) = f(x_{n-1}) = \ldots. = f(f(f(f(x_{n-4}))))\ldots..$$

Chain Rule for differentiation:

$$\frac{df^n}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx_{n-2}} \cdot \frac{dx_{n-2}}{dx} = \ldots\ldots..$$

$$= \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{df(x_{n-2})}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{df(x_{n-2})}{dx_{n-2}} \cdot \frac{df(x_{n-3})}{dx_{n-3}} \cdot \cdot \cdot \cdot \cdot \frac{df(x)}{dx}$$

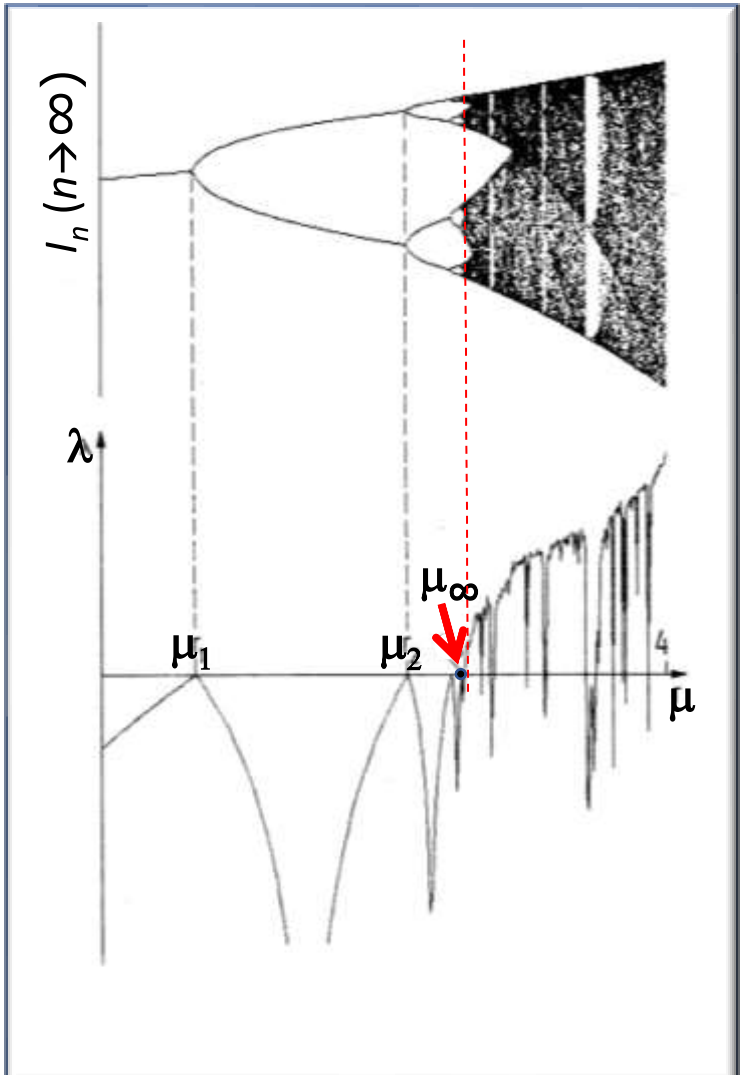
explicit, direct

$$Ln\left|\frac{df^n}{dx}\right| = Ln\prod_{i=0}^{n-1}\left|f'(x_i)\right|_{x_i} = \sum_{i=0}^{n-1} Ln\left|f'(x_i)\right|_{x_i}$$

$$\boxed{\lambda_n = \frac{1}{n} \cdot \sum_{i=0}^{\mathbf{n}-1} Ln\left|f'(x_i)\right|_{x_i}}$$

Large n → test the shape of f at many positions x_i

Lyapunov Exponent = $f(\mu)$



Asymptotic iterates and Lyapunov exponent for the logistic map:

Gain factors μ determine dynamics

$\mu \geq \mu_1$: at least bifurcation

$\mu \geq \mu_2$: at least 2 bifurcations

$\mu \geq \mu_\infty$: λ generally >0, → Chaotic system behavior, small special domains for (relatively) orderly behavior.

Similar :

$$f(x) := \mu \cdot x^k \left(1 - x^k\right)^{1/k} \quad \textit{and}$$

$$f(x) := \mu(x) \cdot x^k \left(1 - x^k\right)^{1/k}$$

Outlook and Conclusions (for our environment)

- Non-linear dynamics of complex systems can lead to orderly or chaotic behavior, depending on non-linearity → amplification μ for log. map. strength of positive feed back loops.

- Chaotic dynamics include sudden wild oscillations in system properties at "Tipping Points,"

- Given an observed non-linear behavior for a specific system (example: Earth albedo), it is possible to estimate a Logistic-Map model amplification parameter μ.

- Extensions of simple 1D Logistic-Map model include multiple dimensions {x,y} provide understanding of population dynamics (predator-prey)

$$dx/dt = \mu(x,y) \cdot x \cdot [1-x] \qquad dy/dt = \mu(x,y) \cdot y \cdot [1-y]$$

- Earth albedo can change rapidly, leading to tipping points in climate.