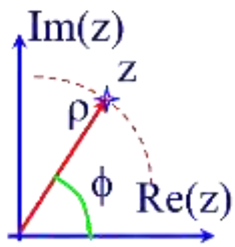
Complex Numbers

Representations of Complex Numbers



There are two ways of representing a complex number as a point in the Complex Plane, in Cartesian or polar coordiantes. The ***Real*** and ***Imaginary*** parts of a complex number z are the components (like$x$ and y) of the number z in a Cartesian coordinate system. One writes

$$z = x+i\cdot y = Re\{z\}+i\cdot Im\{z\} \qquad (1)$$

Here, $$i := \sqrt{(-1)} \qquad (2)$$

is the unit of the ***imaginary numbers***. The quantity

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} \qquad (3)$$

is the norm or absolute value of z.
The number z has a ***complex conjugate*** number z^*, which is related to z by having the negative imaginary part:

$$z^* = (\mathrm{Re}\, z + i\cdot \mathrm{Im}\, z)^* = (\mathrm{Re}\, z - i\cdot \mathrm{Im}\, z) \qquad (4)$$

It is easy to see that, because of $(a+b)\cdot(a-b) = a^2 - b^2$,

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} = \\ = \sqrt{(\mathrm{Re}\, z + i\cdot \mathrm{Im}\, z)\cdot(\mathrm{Re}\, z - i\cdot \mathrm{Im}\, z)} = \sqrt{z\cdot z^*} \qquad (5)$$

Instead of Cartesian coordinates, the point z can be represented by the length $\rho \geq 0$ of a ray from the origin to the number and the "phase angle" ϕ which the ray encloses with the real axis. Then,

$$z = \rho\, e^{i\phi} \tag{6}$$

The equivalence of the representations (1) and (6) requires that

$$\rho = |z| \tag{7}$$

and

$$\phi = \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) \tag{8}$$

Since

$$\operatorname{Re} z = |z| \cdot \cos(\phi) \qquad \operatorname{Im} z = |z| \cdot \sin(\phi) \tag{9}$$

one also derives ***Euler's Formula***

$$e^{i\phi} = \cos(\phi) + i \cdot \sin(\phi) \tag{10}$$

Calculations with Complex Numbers

Addition and subtraction of two complex numbers, z_1 and z_2 are best performed in the Cartesian representation:

$$\begin{aligned} z = z_1 + z_2 &= \operatorname{Re}(z_1 + z_2) + i \cdot \operatorname{Im}(z_1 + z_2) \\ &= \{\operatorname{Re}(z_1) + \operatorname{Re}(z_2)\} + i \cdot \{\operatorname{Im}(z_1) + \operatorname{Im}(z_2)\} \end{aligned} \tag{11}$$

Multiplication with a real number α or another complex number is best performed in the polar representation of Equ. 6:

$$\alpha z = (\alpha \rho) e^{i\phi} \quad (12)$$

$$\begin{aligned} z = \rho e^{i\phi} &= z_1 \cdot z_2 = \left(\rho_1 e^{i\phi_1}\right) \cdot \left(\rho_2 e^{i\phi_2}\right) = \\ &= \left(\rho_1 \cdot \rho_2\right) \cdot \left\{e^{i\phi_1} \cdot e^{i\phi_2}\right\} = \left(\rho_1 \cdot \rho_2\right) \cdot e^{i(\phi_2 + \phi_1)} \end{aligned} \quad (13)$$

In other words, in a multiplication of two numbers, their absolute values multiply, while their phases add. The absolute value of a "phase factor" with $\rho = 1$ is

$$|z| = |e^{i\phi}| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} = \sqrt{(\cos(\phi))^2 + (\sin(\phi))^2} = 1 \quad (14)$$